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#### EFFECTIVE-FIELD METHOD IN THE STATICS OF COMPOSITE MATERIALS

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In this paper we study a linearly elastic composite medium consisting of a uniform matrix containing a random number of inclusions which have an arbitrary shape and nonuniform bulk mechanical properties. The classical problem [1-3] of estimating the effective moduli and average stress concentrators on inclusions is solved. The approach proposed in this paper is an extension of the effective-field method (EFM), presented in [4-6] for the case when the mechanical properties of the matrix are the same as those of the comparison medium. The generalized EFM includes as particular cases the well-known methods of structural mechanics: the effective-medium method [3], the generalized singular approximation method [3], the conditional moment method [7, 8], the Mori-Tanaka-Eshelby method [9, 10], and methods based on variational principles [2].

1. General Equations. Consider a macroscopic region  $w$  with characteristic function  $W$  and containing a random set  $X = (V_k, x_k, \omega_k)$  of ellipsoids  $v_k$  with characteristic functions  $V_k$  and centers  $x_k$ , forming a Poisson set, semiaxes  $a_k^i (a_k^1 \geq a_k^2 \geq a_k^3)$ , and Euler angles  $\omega_k$ . The local equation of state of the material, relating the stress tensor  $\sigma(x)$  and the strain tensor  $\varepsilon(x)$ , is given in the form

$$\sigma(x) = L(x)\varepsilon(x), \quad (1.1)$$

where  $L(x)$ , which is a tetravalent tensor of the elastic moduli, is assumed to be homogeneous in the matrix  $v_0 = w \setminus v (v = \bigcup_{h=1} v_h)$ :  $L(x) = L^{(0)}$  in each inclusion  $v_k$ , where  $k = 1, 2, \dots$ ,

and  $L(x) = L^{(k)}(x)$  is, generally speaking, an inhomogeneous function of the coordinates. Substituting Eq. (1.1) into the equation of equilibrium with given boundary conditions on the displacements  $u(x)$ , we obtain a differential equation for the displacements:

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$$\nabla L_c \nabla u = -\nabla L_1(x) \nabla u. \quad (1.2)$$

Here  $\nabla$  is the symmetrized gradient operator and  $L_1(x) \equiv L(x) - L_c$ . We introduce the homogeneous tensor of elastic moduli  $L_c = \text{const}$  of the comparison medium; in the general case,  $L_c \neq L^{(0)}$ . Reducing Eq. (1.2) to an integral equation and transforming it according to the scheme of [4-6], we find

$$\varepsilon(x) = \varepsilon_0 + \int U(x-y) [L_1(y) \varepsilon(y) - \langle L_1 \varepsilon \rangle] dy, \quad (1.3)$$

where  $U(x-y) = \nabla \nabla G(x-y)$ ;  $G$  is the Green's tensor of the Lamé equation for an unbounded comparison medium with modulus  $L_c$ :  $\nabla L_c \nabla G(x) = -I \delta(x)$ ; and  $\delta(x)$  is a delta function. In Eq. (1.3) and below  $\langle (\cdot) \rangle$ ,  $\langle (\cdot) \rangle_{x_1, \dots, x_n}$ ;  $x_{n+1}, \dots, x_m$  denote the average and conditional average over an ensemble of statistically homogeneous and ergodic field  $X$  under the condition that inclusion are located at the points  $x_1, \dots, x_m \neq x_{n+1}, \dots, x_m$  and  $\varepsilon_0 \equiv \langle \varepsilon \rangle$ . We also define the average over the volume of the component as

$$\langle (\cdot) \rangle_\alpha = \bar{v}_\alpha \int (\cdot) V_\alpha(x) dx, \quad \bar{v}_\alpha = \text{mes } v_\alpha \quad (\alpha = 0, 1, \dots). \quad (1.4)$$

In the derivation of Eq. (1.3) it was assumed that the region  $w$  contains a statistically large number of inclusions  $v$ ; all random quantities considered below are described by statistically homogeneous random fields, so that the averaging over an ensemble can be replaced by averaging over the volume; the distance  $\rho = \rho(x)$  as a function of  $x$  from the boundary  $\partial w$  of the region  $w$  is much greater than the characteristic size of the inclusions  $a^{1/\rho} \ll 1$ . For this reason, the subsequent solution of the problem (1.3) is of zero-order accuracy with respect to the small parameter  $a^{1/\rho}$ .

In order to estimate the effective moduli we average the local equation (1.1) over the volume  $\langle \sigma \rangle = L^{(0)} \langle \varepsilon \rangle + \langle (L(x) - L^{(0)}) \varepsilon(x) V(x) \rangle$ . Then

$$L^* = L^{(0)} + B^*, \quad \langle L_1 \varepsilon V \rangle - L_1^{(0)} \langle \varepsilon V \rangle \equiv B^* \langle \varepsilon \rangle \quad (1.5)$$

$$\left( V \equiv \sum_{i=1} V_i, \quad L_1^{(\alpha)} = L^{(\alpha)} - L_c, \quad \alpha = 0, 1, \dots \right).$$

Thus, in order to find the effective parameters we must estimate the average value of the polarization tensor inside the inclusions  $\langle (L(x) - L^{(0)}) \varepsilon V(x) \rangle$ . Equation (1.3) is much easier to solve when the deformation fields are studied only inside the inclusion. There are two fundamentally different approaches to ensuring that the integration on the right-hand side of Eq. (1.3) extends only over the volume of the inclusions. In the first one we postulate  $L_c \equiv L^{(0)}$ . Then  $L_1^{(0)} \equiv 0$  and we write Eq. (1.3) as

$$\varepsilon(x) = \varepsilon_0 + \int U(x-y) [L_1(y) \varepsilon(y) V(y) - \langle L_1 \varepsilon V \rangle] dy. \quad (1.6)$$

In the second one we choose  $L_c$  quite arbitrarily, but we make the additional assumption that the deformation fields in the matrix are homogeneous:  $\varepsilon(x) \equiv \langle \varepsilon \rangle_0$ ,  $x \in v_0$ . Then Eq. (1.3) is equivalent to the equation

$$\varepsilon(x) = \varepsilon_0 + \int U(x-y) [(L_1(y) \varepsilon(y) - L_1^{(0)} \langle \varepsilon \rangle_0) V(y) - \langle (L_1 \varepsilon - L_1^{(0)} \langle \varepsilon \rangle_0) V \rangle] dy. \quad (1.7)$$

We introduce the notation  $M_1(y) \equiv L_1(y) V(y)$ ,  $\alpha(y) \equiv -L_1^{(0)} \langle \varepsilon \rangle_0 V(y)$  and represent Eq. (1.7) in the unified form

$$\varepsilon(x) = \varepsilon_0 + \int U(x-y) \{M_1(y) \varepsilon(y) + \alpha(y) - \langle M_1 \varepsilon + \alpha \rangle\} dy, \quad (1.8)$$

and in addition for  $L_c \equiv L^{(0)}$  in Eq. (1.8) there exists  $\alpha(y) \equiv 0$ .

In order to calculate the averages  $\langle M_1 \varepsilon \rangle$  in Eq. (1.8), which are required in order to estimate the effective modulus  $L^*$ , we introduce  $\varphi(v_m | v_1, \dots, v_n)$  - the conditional distribution

function of the m-th inclusion in the region with fixed inclusions in the regions  $v_1, \dots, v_n$ . As far as the functions  $\varphi(v_m|v_1, \dots, v_n)$  are concerned, it is known that  $\varphi(v_m|v_1, \dots, v_n) = 0$  for  $x_m$  lying inside some correlation well, consisting of the union of the regions  $v_j^0 \supset v_j$  ( $j = 1, \dots, n$ ) with characteristic functions  $V_j^0$ , and  $\varphi(v_m|v_1, \dots, v_n) \rightarrow \varphi(v_m)$  as  $|x_i - x_m| \rightarrow \infty$ ,  $i = 1, \dots, n$ . We average Eq. (1.8) on the sets  $X(\cdot|v_1)$ ,  $X(\cdot|v_1, v_2)$  with fixed inclusions  $v_1$ ;  $v_1$  and  $v_2$ , and so on with the help of different distribution functions  $\varphi(v_m|v_1, \dots, v_n)$ . This gives an infinite system of coupled integral equations:

$$\begin{aligned} \varepsilon(x) - \int U(x-y) V_1(y) \langle M_1(y) \varepsilon(y) + \alpha(y) | x_1 \rangle dy = \\ = \varepsilon_0 + \int U(x-y) [\langle M_1(y) \varepsilon(y) + \alpha(y) | y; x \rangle - \langle M_1 \varepsilon + \alpha \rangle] dy, \\ \varepsilon(x) - \sum_{i=1}^n \int U(x-y) V_i(y) \langle M_1(y) \varepsilon(y) + \alpha(y) | x_1, \dots, x_n \rangle dy = \\ = \varepsilon_0 + \int U(x-y) [\langle M_1(y) \varepsilon(y) + \alpha(y) | y; x_1, \dots, x_n \rangle - \langle M_1 \varepsilon + \alpha \rangle] dy. \end{aligned} \quad (1.9)$$

Since  $x$  in the n-th row of system (1.9) can run through the values in the inclusions  $v_1, \dots, v_n$ , the n-th row actually contains n equations. We designate the right-hand side of the n-th ( $n = 1, 2, \dots$ ) row by the field  $\hat{\varepsilon}(x)_1, \dots, n$ , which physically is simply the formation field in which the fixed n inclusions are located. Each inclusion  $v_i$  from the chosen fixed inclusions is located in the field

$$\bar{\varepsilon}_i(x) = \hat{\varepsilon}(x)_{1, \dots, n} + \sum_{j \neq i} \int U(x-y) V_j(y) [M_1(y) \sigma(y) + \alpha(y)] dy, \quad x \in v_i. \quad (1.10)$$

As follows from the structure of Eqs. (1.9) and (1.10), the stress fields in the inclusions  $v_i$  depend only on, generally speaking, the inhomogeneous field  $\bar{\varepsilon}_i$  in the region  $v_i$ . In order to be able to neglect below the dependence of the terms in system (1.9) on  $x \in v_i$ , we average each n-th row ( $n = 1, 2, \dots$ ) over the volume of the i-th inclusion ( $i = 1, \dots, n$ ), and then

$$\begin{aligned} \langle \varepsilon | x_1, \dots, x_n \rangle_i - \sum_{j=1}^n \bar{v}_i^{-1} \int \int U(x-y) V_i(x) V_j(y) \langle M_1(y) \varepsilon(y) + \\ + \alpha(y) | x_1, \dots, x_n \rangle dy dx = \varepsilon_0 + \bar{v}_i^{-1} \int \int U(x-y) V_i(x) V_j(y) [\langle M_1(y) \varepsilon(y) + \\ + \alpha(y) | y; x_1, \dots, x_n \rangle - \langle M_1 \varepsilon + \alpha \rangle] dy dx. \end{aligned} \quad (1.11)$$

Under the assumptions made about the homogeneity of the tensors  $\varepsilon_0, L^{(0)}$ , and  $L_C$  and the statistical homogeneity and ergodicity of the field, system (1.11) is exact only for  $L_C = L^{(0)}$ . Then  $a(y) \equiv 0$ . For  $L_C \neq L^{(0)}$ , system (1.11) is obtained under the additional assumption that the deformation field in the matrix is homogeneous. In the derivation of Eq. (1.11) no restrictions were imposed on the shape and mechanical properties of the inclusions and the structure of the conditional distribution function  $\varphi(v_j|v_1, \dots, v_n)$ .

2. The Effective Field. In order to close system (1.11) and solve it approximately, we adopt the hypotheses of the effective-field method [4, 5]:

H1) every inclusion  $v_i$  is an ellipsoid, can be approximated by a point when analyzing the stress fields outside it, and is located in a homogeneous field  $\bar{\varepsilon}(x_i)$ ;

H2) for sufficiently large n we have the closure  $(\hat{\varepsilon}(x)_{1, \dots, j, \dots, n+1})_i = \langle \hat{\varepsilon}(x)_{1, \dots, n+1} \rangle_i$ , where the right-hand side of the equality does not contain the index  $j \neq i$ ,  $1 \leq j \leq n$  and  $x \in v_i$ .

For ellipsoidal inclusions  $v_i$  we obtain from Eqs. (1.10) and (1.11) the algebraic equation

$$\langle \varepsilon(x) \rangle_i - \langle U(x) \rangle_i \langle M_1(x) \varepsilon(x) + \alpha(x) \rangle_i = \langle \bar{\varepsilon}(x) \rangle_i, \quad (2.1)$$

in which, according to a property of potentials inside and ellipsoid [1, 11], the tensor  $\langle U(x) \rangle_i$  for  $x \in v_i$ . The assumption that inclusions are ellipsoidal can be weakened; this will be demonstrated in Sec. 7. In order to solve system (1.11) it is necessary to know how  $\langle \varepsilon(x) \rangle_i$ ,  $\langle M_1(x) \varepsilon(x) + \alpha(x) \rangle_i$  depend on  $\langle \bar{\varepsilon}(x_i) \rangle$ . Since problem (2.1) is linear, there exist constant tensors  $A_i$  and  $C_i$  of rank 4, such that

$$\begin{aligned} \langle \varepsilon(x) \rangle_i &= A_i \langle \bar{\varepsilon}(x_i) \rangle_i + C_i \alpha(x_i), \\ \bar{v}_i \langle M_1(x) \varepsilon(x) + \alpha(x) \rangle_i &= R_i \langle \bar{\varepsilon}(x_i) \rangle_i + F_i \alpha(x_i) \\ (R_i = \langle U(x) \rangle_i^{-1} (A_i - I) \bar{v}_i, \quad F_i = \langle U(x) \rangle_i^{-1} C_i \bar{v}_i). \end{aligned} \quad (2.2)$$

For example, for a homogeneous ellipsoidal inclusion  $v_i$  with  $M_1^{(i)} = \text{const}$

$$A_i = (I + P_i M_1^{(i)})^{-1}, \quad C_i = -A_i P_i. \quad (2.3)$$

Here  $P_i = - \int U(x-y) V_i(y) dy$  ( $x \in v_i$ ) is a constant tensor, which does not depend on the elastic moduli and dimensions (but not the shape) of the ellipsoid  $v_i$ ; the rules for calculating  $P_i$  in Eq. (2.3) for different cases of anisotropy of the shape of inclusions and properties of the matrix are examined in [1]. The tensors  $A_i$  and  $C_i$  can be found, for example, numerically for any structure of the field  $\bar{\varepsilon}(x)$  ( $x \in v_i$ ) and they depend on the structure of the field. In what follows, in order to obtain understandable results we assume that the field  $\bar{\varepsilon}(x)$  is weakly inhomogeneous and homogeneous according to the hypothesis H1 inside any region  $v_i$ :  $\bar{\varepsilon}(x) = \bar{\varepsilon}(x_i)$ ,  $x \in v_i$ . In the case of a homogeneous field  $\bar{\varepsilon}(x)$  ( $x \in v_i$ ) problem (2.2) has been solved analytically for a layered ellipsoid [12] and a layered sphere [13].

We interpret the approximation of point-like inclusions in hypothesis H1 as meaning that the following equation is satisfied:

$$\begin{aligned} \int U(x-y) V_i(y) (M_1(y) \varepsilon(y) + \alpha(y)) dy = \\ = \langle U(x-y) \rangle_i \langle M_1(y) \varepsilon(y) + \alpha(y) \rangle_i \bar{v}_i \end{aligned} \quad (2.4)$$

for  $x \notin v_i$ . The relation (2.4) means that the asymptotic behavior of the disturbed field of an inclusion of finite size is identical at infinity to the asymptotic behavior of a point inclusion which simulates it.

3. Estimate of the Interaction of a Finite Number of Inclusions. Under the hypothesis H1, system (1.11) with fixed values  $(\hat{\varepsilon}(x)_{1, \dots, n})_i$  ( $x \in v_i$ ) of the right-hand sides of the equations becomes algebraic:

$$\begin{aligned} \langle \bar{\varepsilon}(x) | x_1, \dots, x_n \rangle_i - \sum_{j=1}^n \bar{v}_i^{-1} \int \int U(x-y) V_j(y) V_i(x) \langle M_1(y) \varepsilon(y) + \\ + \alpha(y) | x_1, \dots, x_n \rangle_j dx dy = \langle \hat{\varepsilon}(x)_{1, \dots, n} \rangle. \end{aligned} \quad (3.1)$$

Using Eq. (2.2) for a single inclusion in the field  $\langle \hat{\varepsilon}(x_i) \rangle_i$  (1.10) ( $i = 1, \dots, n$ ), we rewrite Eq. (3.1) in the form

$$\begin{aligned} \langle \bar{\varepsilon}(x) | x_1, \dots, x_n \rangle_i - \sum_{j \neq i} (\bar{v}_j \bar{v}_i)^{-1} \int \int U(x-y) V_j(y) V_i(x) \times \\ \times \{ R_j \langle \bar{\varepsilon}(y) | x_1, \dots, x_n \rangle_j + F_j \alpha \} dx dy = \langle \hat{\varepsilon}(x)_{1, \dots, n} \rangle. \end{aligned} \quad (3.2)$$

System (3.2) is linear algebraic with respect to  $\langle \hat{\varepsilon}(x) | x_1, \dots, x_n \rangle \equiv \langle \hat{\varepsilon}(x_i) | x_1, \dots, x_n \rangle$  and can be solved by the standard methods of linear algebra. For this we change from the tensor form of system (3.2) to a matrix form [3]. We construct the matrix  $Z^{-1}$  with elements  $Z_{mk}^{-1}$  ( $m, k = 1, \dots, n$ ) in the form of  $6 \times 6$  submatrices

$$\begin{aligned} Z_{mk}^{-1} &= I \delta_{mk} - (1 - \delta_{mk}) R_m S(x_m - x_k), \\ S(x_m - x_k) &= (\bar{v}_m \bar{v}_k)^{-1} \int \int U(x-y) V_m(x) V_k(y) dx dy, \end{aligned}$$

and then represent the solution of system (3.2) as

$$R_i \langle \bar{\varepsilon}(x_i) | x_1, \dots, x_n \rangle + F_i \alpha = \sum_{j=1}^n Z_{ij} (R_j \langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_j + F_j \alpha). \quad (3.3)$$

The solution of system (3.2) can also be constructed by the method of successive approximations [4, 5]. Then, taking into account the first two iterations, we have

$$R_i \langle \bar{\varepsilon}(x_i) | x_1, \dots, x_n \rangle + F_i = R_i \langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_i + F_i + \sum_{j \neq i} R_i S(x_i - x_j) [R_j \langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_j + F_j]. \quad (3.4)$$

We note that the use of the "quasicrystalline" approximation [14]

$$\langle \varepsilon(x) | x_1, \dots, x_n \rangle_i = \langle \varepsilon(x) \rangle_i \quad (3.5)$$

is equivalent to the assumptions

$$\langle \bar{\varepsilon}(x_i) | x_1, \dots, x_n \rangle = \langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_i, \quad Z_{ij} = I \delta_{ij}. \quad (3.6)$$

Particular cases of formulas (3.3) are examined for  $L_c = L^{(0)}$  in [4, 15-18] for two spherical inclusions [15, 16] and flat spheroidal cracks [17, 18]. It is shown that for two equal circular cracks in a plane, which lie in a normally loaded plane, the assumption that the fields  $\bar{\varepsilon}(x_i)$  are homogeneous (hypothesis H1) near a crack leads to an error of 2% in the estimated stress intensity coefficient with the distance between cracks equal to 0.01 of their radius [18]. The high accuracy of the effective-field model of deformations or stresses [4, 17, 19] (called a pseudoload in [19] is a result of the fact that the field  $\bar{\varepsilon}(x)(x \in v_i)$  within an inclusion changes insignificantly; the error of the approximation  $\bar{\varepsilon}(x)(x \in v_i)$  by polynomials of different degree is estimated in [17].

4. Estimate of the Effective Modulus. The solutions obtained for one inclusion (2.2) and a finite number of inclusions (3.3), which are located in effective fields  $\bar{\varepsilon}(x)$  and  $\widehat{\varepsilon}(x)_{1, \dots, n}$ , and the adoption of the hypothesis H2 make it possible to solve system (1.11). Indeed, from Eq. (1.11) we have a closed system of integral equations for the fields  $\langle \widehat{\varepsilon}(x_i)_{1, \dots, j} \rangle_i$  ( $j = 1, \dots, n$ ;  $i = 1, \dots, j$ ):

$$\begin{aligned} \langle \widehat{\varepsilon}(x)_{1, \dots, j} \rangle_i &= \\ &= \varepsilon_0 + \int \left\{ S(x_i - x_q) V(x_q; x_1, \dots, x_j) \sum_{l=1}^{j+1} Z_{ql} (R_l \langle \widehat{\varepsilon}(x)_{1, \dots, j+1} \rangle_l + F_l \alpha) - \right. \\ &\quad \left. - S_i(x_i - x_q) \langle R \widehat{\varepsilon}_i + F \alpha \rangle \right\} dx_q, \\ \langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_i &= \\ &= \varepsilon_0 + \int \left\{ S(x_i - x_q) V(x_q; x_1, \dots, x_{n-1}) \sum_{l=1}^n Z_{ql} (R_l \langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_l + F_l \alpha) - \right. \\ &\quad \left. - S_i(x_i - x_q) \langle R \widehat{\varepsilon}_i + F \alpha \rangle \right\} dx_q, \end{aligned} \quad (4.1)$$

where  $S_i(x_i - x_q) = \bar{v}_i^{-1} \int U(x - x_q) V_i(x) dx$ ,  $x_q \notin v_i$ . The tensor  $\widehat{\varepsilon}_{1, \dots, n}$  on the right-hand side of the last equation of (4.1) is formed from the tensor  $\widehat{\varepsilon}_{1, \dots, n}$  on the left-hand side by replacing one of the indices by  $q$ . System (4.1) is linear with respect to  $\langle \widehat{\varepsilon}(x)_{1, \dots, j} \rangle_{\ell}$  ( $j = 1, \dots, n$ ;  $\ell = 1, \dots, j$ ), and each  $j$ -th row with  $\langle \widehat{\varepsilon}(x)_{1, \dots, j} \rangle_{\ell}$  on the left-hand side contains  $j$  equations, since  $i = 1, \dots, j$ . The value of  $\langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_i$  ( $i = 1, \dots, n$ ) is estimated from the last row in Eq. (4.1) by the method of successive approximations with all possible positions of the inclusions  $v_1, \dots, v_n$ . It is also necessary to take into account the fact that  $\langle \widehat{\varepsilon}(x)_{1, \dots, n} \rangle_i \rightarrow \langle \bar{\varepsilon}(x_i) \rangle_i$  as  $|x_j - x_i| \rightarrow \infty$ ,  $j = 1, \dots, n$ ,  $j \neq i$ . We substitute the value found for  $\langle \bar{\varepsilon}(x)_{1, \dots, n} \rangle_i$  into the right-hand side of the  $(n-1)$ -th row of system (4.1), determine  $\langle \widehat{\varepsilon}(x)_{1, \dots, n-1} \rangle_i$  ( $i = 1, \dots, n-1$ ) and so on. After estimating  $\langle \widehat{\varepsilon}(x_i) \rangle_i$  and  $\langle M_1(x) \varepsilon(x) + \alpha(x) \rangle_i$ , in the relations (1.11) and (2.2) we estimate from formula (1.7) the effective modulus  $L^*$  with the help of the equality

$$\langle \varepsilon \rangle_0 = (c_0 + \langle VC \rangle L_1^{(0)})^{-1} (\varepsilon_0 - \langle VA \widehat{\varepsilon}_1 \rangle), \quad c_\alpha = \langle V_\alpha \rangle \quad (\alpha = 0, 1, \dots). \quad (4.2)$$

We now estimate, at the physical level of rigorosity, the lower limit of the accuracy of the proposed EFM. In [15, 20], in analyzing the equations of the theory of elasticity of composite materials with  $L_c = L^{(0)}$ , analogous to Eqs. (1.9), it was assumed that  $\langle \widehat{\varepsilon}(x)_{12} \rangle_i = \varepsilon_0$  ( $i = 1, 2$ ),  $\langle \widehat{\varepsilon}(x)_1 \rangle_1 = \varepsilon_0$ . Then  $\langle M_1(x_2) \varepsilon(x_2) | x_2; x_1 \rangle$  was estimated from the second equation in Eqs. (1.1) and the effective parameters (1.5) were estimated from the first equation in Eqs. (1.9) and (2.2). The assumption  $\langle \widehat{\varepsilon}(x)_1 \rangle_1 = \varepsilon_0$  makes it possible to determine the coefficient of the first power of the concentration  $c$  of inclusions as a function of  $L^* = L^*(c)$ , and the assumption  $\langle \varepsilon(x)_{12} \rangle_i = \varepsilon_0$  ( $i = 1, 2$ ) makes it possible to determine the coefficient of  $c^2$ . In [20] the problem of estimating the pair interaction of different spherical inclusions  $\langle M_1(x_2) \varepsilon(x_2) | x_2; x_1 \rangle$  with  $\langle \widehat{\varepsilon}(x)_{12} \rangle_i = \varepsilon_0$  ( $i = 1, 2$ ) was solved numerically with the help of Legendre polynomials, and in [15] it was solved under an assumption that is stronger than (2.4):

$$\int U(x-y) V_2(y) M_1(y) \varepsilon(y) dy = U(x_1 - x_2) M_1(x_2) \varepsilon(x_2), \quad x \in v_1. \quad (4.3)$$

It was found that for hard spherical inclusions in an incompressible matrix the coefficient of  $c^2$ , according to the data of [15], is equal to 4.84 and differs from the more accurate numerical result 5.01 [20] by 3.3%. Similarly, the assumption  $\langle \widehat{\varepsilon}(x)_1, \dots, n \rangle = \varepsilon_0$ , instead of  $\langle \widehat{\varepsilon}(x)_{12} \rangle_i = \varepsilon_0$  ( $i = 1, 2$ ), makes it possible to represent the dependence  $L^* = L^*(c)$  in the form of a polynomial of degree  $n$  in  $c$ . Since in the EFM the field  $\langle \widehat{\varepsilon}(x)_1, \dots, n \rangle_i$  is not postulated but rather estimated from the self-consistent Eqs. (4.1), the EFM in the solution of the problem of interaction and inclusions (3.3) and (4.1), gives an accuracy of the estimate  $L^* = L^*(c)$  higher than the degree of the polynomial  $n$ . We call the solution of Eqs. (3.3) and (4.1) the solution of Eq. (1.10) of the  $n$ -particle approximation. In the solution of the  $n$ -particle approximation problem one studies not the entire space, but rather a finite region  $v(x_1, \dots, x_n) \supset v_1, \dots, v_n$  (which, generally speaking, depends on the number and dimensions of the inclusions  $v_1, \dots, v_n$ ), since the value of the integral on the right-hand side of Eq. (4.1) over the region  $w \setminus v(x_1, \dots, x_n)$  becomes negligibly small. For example, in solving problem (1.8) of the two-particle approximation the spherical region  $v(x_1, x_2)$  centered at  $x_1$  and having a radius that is five times greater than the radius of the inclusions gives an error in estimating  $L^*$  of not more than 3% as compared with integration of Eq. (4.1) with  $n = 2$  over the region  $w$ . Thus, there exists a locality principle [21] and the region  $v(x_1, \dots, x_n)$  is finite. The Equality  $v(x_1, \dots, x_n)$  is thereby satisfied asymptotically for large  $n$  and the hypothesis H2 is justified. In order to approximate the field  $\langle \widehat{\varepsilon}(x)_1, \dots, n+1 \rangle_i = \langle \widehat{\varepsilon}(x)_1, \dots, n \rangle_i$ , according to the hypothesis H2 the index  $j$  for which  $\max |x_j - x_1|$ ,  $j = 1, \dots, n+1$ .

5. Analytical Estimate of  $L^*$ . The solution of problem (4.1) on a cell of the  $n$ -th approximation presupposes a solution of system (4.1) for arbitrary coordinates of the centers of the inclusions and arbitrary orientations of the inclusions. The two-particle approximation and the assumption

$$\langle \widehat{\varepsilon}(x)_{12} \rangle_i = \langle \widehat{\varepsilon}(x_i) \rangle = \text{const} \quad (i = 1, 2). \quad (5.1)$$

greatly simplify the problem. Then we obtain from the first equation in Eqs. (4.1) and (3.3)

$$\begin{aligned} R_i \langle \widehat{\varepsilon}_i \rangle + F_i \alpha &= (R_i \varepsilon_0 + F_i \alpha) + R_i \int \left\{ S(x_i - x_j) \sum_{l \neq i, j} Z_{jl} (R_l \langle \widehat{\varepsilon}_l \rangle + \right. \\ &\left. + F_l \alpha) \varphi(v_j | v_j; v_i) - S_i(x_i - x_j) \langle R \widehat{\varepsilon} + F \alpha \rangle_{n_j} \right\} dx_j. \end{aligned} \quad (5.2)$$

System (5.2) is a linear algebraic system and can be solved for an arbitrary number of components, under the assumption that the inclusions refer to different components, if the mechanical properties, sizes, or orientations are different. The number of components and hence also the dimension of system (5.2) can be reduced by two to three orders of magnitude by postulating that the fields  $\widehat{\varepsilon}_1$  are independent of the orientation of the inclusions  $v_1$ . Then, averaging Eq. (5.2) over the orientations of the inclusions  $v_1$  with the help of the operation  $\langle (\cdot) \rangle_\omega$  and assuming, in order to obtain visible results, that  $\langle R_i S_i \rangle_\omega = \langle R_i \rangle_\omega \langle S_i \rangle_\omega$ ,  $\langle R_i S_{j\ell} \rangle_\omega$

$R_{\ell} \rangle_\omega = \langle R_i S_{j\ell} R_{\ell} \rangle_\omega = \langle R_i \rangle_\omega \langle S_{j\ell} \rangle_\omega \langle R_{\ell} \rangle_\omega$ , we have

$$\begin{aligned}
\langle R_i \rangle_\omega \langle \bar{\varepsilon}_i \rangle + \langle F_i \rangle_\omega \alpha &= \langle R_i \rangle_\omega \varepsilon_0 + \langle F_i \rangle_\omega \alpha + \\
+ \langle R_i \rangle_\omega \int \langle S(x_i - x_j) \rangle_\omega \sum_{l \neq i, j} Z_{jl} (\langle R_l \rangle_\omega \langle \bar{\varepsilon}_l \rangle_\omega + \\
+ \langle F_l \rangle_\omega \alpha) \varphi(v_j | v_j; v_i) - \langle S_i \rangle_\omega \langle \bar{R} \varepsilon + F \alpha \rangle dx_j,
\end{aligned} \tag{5.3}$$

where in calculating  $Z_{ij}$   $R_i S(x_i - x_j)$  was replaced by  $\langle R_i \rangle_\omega \langle S(x_i - x_j) \rangle_\omega$ .

Equation (5.3) can be represented in the matrix form

$$\sum_{j=1}^N Y_{ij} (\langle R_j \rangle_\omega \langle \bar{\varepsilon}_j \rangle_\omega + \langle F_j \rangle_\omega \alpha) = (\langle R_i \rangle_\omega \varepsilon_0 + \langle F_i \rangle_\omega \alpha) \quad (i = 1, \dots, N). \tag{5.4}$$

Here the submatrices are

$$\begin{aligned}
Y_{ij} = \delta_{ij} \left( I - \langle R_i \rangle_\omega \sum_{k=1}^N \int \langle S(x_i - x_k) \rangle_\omega Z_{ki} \varphi(v_k | v_k; v_i) dx_k \right) - \\
- \langle R_i \rangle_\omega \int \{ \langle S(x_i - x_j) \rangle_\omega Z_{ji} \varphi(v_j | v_j; v_i) - \\
- \langle S_i(x_i - x_j) \rangle_\omega n_j \} V(x_j; x_i) dx_j - \langle R_i \rangle_\omega \langle P(v'_{ij}) \rangle_\omega n_j.
\end{aligned} \tag{5.5}$$

In the case of the "quasicrystalline" approximation (3.5) and (3.6) expression (5.5) is simplified:

$$\begin{aligned}
Y_{ij} = I \delta_{ij} - \langle R_i \rangle_\omega \langle P(v'_{ij}) \rangle_\omega n_j - \\
- \langle R_i \rangle_\omega \int (\langle S(x_i - x_j) \rangle_\omega \varphi(v_j | v_j; v_i) - \langle S_i \rangle_\omega n_j) V(x_j; x_i) dx_j.
\end{aligned} \tag{5.6}$$

We obtain from Eqs. (5.4) and (5.5) the solution of (5.3) as

$$\langle R_i \rangle_\omega \langle \bar{\varepsilon}_i \rangle + \langle F_i \rangle_\omega \alpha = \sum_{j=1}^N (Y^{-1})_{ij} (\langle R_j \rangle_\omega \varepsilon_0 + \langle F_j \rangle_\omega \alpha) \tag{5.7}$$

and with the help of Eq. (4.2) we find the following expression for the effective modulus (1.5):

$$\begin{aligned}
L^* = L_c + \sum_{i,j=1}^N n_i Y_{ij}^{-1} \langle R_j \rangle_\omega + \\
+ \left\{ \sum_{i,j=1}^N n_i [Y_{ij}^{-1} \langle F_j \rangle_\omega - \langle F_i \rangle_\omega] \right\} L_1^{(0)} (c_0 - \langle VC \rangle L_1^{(0)})^{-1} \times \\
\times \left[ I + \sum_{i=1}^N c_i \langle A_i \rangle_\omega \langle R_i \rangle_\omega^{-1} \left( \langle F_i \rangle_\omega - \sum_{j=1}^N n_j Y_{ij}^{-1} \langle F_j \rangle_\omega \right) L_1^{(0)} (c_0 - \langle VC \rangle L_1^{(0)})^{-1} \right]^{-1} \times \\
\times \left[ I - \sum_{i=1}^N c_i \langle A_i \rangle_\omega \langle R_i \rangle_\omega^{-1} \sum_{j=1}^N Y_{ij}^{-1} \langle R_j \rangle_\omega \right].
\end{aligned} \tag{5.8}$$

**6. Consequences of the EFM.** We shall show that many of the best known methods of structural mechanics follow from the EFM. We start with the case  $L_c = L^{(0)}$ , when  $\alpha(y) \equiv 0$  and there is no need to postulate that the field of deformation is homogeneous. Then formula (5.8) is simplified substantially:

$$L^* = L^{(0)} + \sum_{i,j=1}^N n_i Y_{ij}^{-1} \langle R_j \rangle_\omega. \tag{6.1}$$

The relations given in [4-6] follow from Eq. (6.1) under additional assumptions about the homogeneity of the inclusion; in turn, a more specific result for identical spherical inclusions, later obtained independently in [22], follows from [4-6]. In the approximate variant of the EFM [23-26] a "quasicrystalline" approximation of the field  $\bar{\varepsilon}(x_1 | x_1; x_2) = \bar{\varepsilon}(x_1)$ , or equivalently,  $Z_{ij} = I \delta_{ij}$  (3.6), was used; this makes it possible to close the first equation of Eqs. (4.1) and limit the solution of Eq. (1.8) to the single-particle approximation. Then,

if it is assumed in addition that the inclusions are homogeneous and  $\varphi(v_j|v_j; v_i)$  depends only on  $|x_i - x_j|$  in the representation of  $Y_{ij}$  (5.6), then we obtain the results of [26]. If, in addition, averaging over the orientations of the inclusions is not studied in Eq. (5.6), then formulas (6.1) are equivalent to those proposed in [23, 24]. In [4-6] it is shown that the two-particle approximation of the solution of problem (4.1) under the assumption (5.1) led, in a number of cases, to estimates of  $L^*$  which are more than two times more accurate than [23, 25].

We note that the initial Eqs. in [24, 25, 27] differed from the exact equation (1.3):

$$\varepsilon(x) = \varepsilon_0 + \int U(x-y) L_1(y) \varepsilon(y) dy, \quad (6.2)$$

which is correct only for a finite number of inclusions. Since the integral in Eq. (6.2) diverges at infinity, the form of  $w$  must be postulated [25] or the action of the generalized function  $U$  on constants  $m = \text{const}$  must be determined [24]:

$$\int U(x-y) m dy = 0, \quad (6.3)$$

Formula (6.3) has not appeared previously in the theory of generalized functions. In the multipole expansion method [27, 28] a particular case of Eqs. (3.4) and (5.2) was used, and in the expression analogous to Eq. (5.2) the term  $S_1(x_i - x_j) \langle R\bar{\varepsilon} + F\alpha \rangle n_j$ , did not appear, which also is incorrect.

The Mori-Tanaka-Eshelby method [9, 29] (references in [10]) is widely used. In this method the mean deformation field in homogeneous inclusions is estimated from the single-particle problem (2.2) under a stronger assumption than (3.6)  $\langle \varepsilon(x_i) \rangle_i = \langle \varepsilon \rangle_0$ :

$$\langle \varepsilon(x) \rangle_i = A_i \langle \varepsilon \rangle_0.$$

Then the identity  $\langle \varepsilon V \rangle - c_0 \langle \varepsilon \rangle_0 = \langle \varepsilon \rangle$  and Eq. (1.7) imply

$$L^* = L^{(0)} + \langle L_1 A V \rangle [I - (\langle A V \rangle - c)]^{-1},$$

which for identically oriented inhomogeneities is a particular case of formulas (5.6) and (6.1). Thus the Mori-Tanaka-Eshelby method [9, 10] and the single-particle approximation of the EFM [23, 24], which are equivalent methods, gave results which were in part duplicated.

We now study the case  $L_C \neq L^{(0)}$ . There is no a priori justification for the specific choice of  $L_C$ , not counting the condition that the quadratic form  $L(L_1 \varepsilon) \varepsilon$ , employed in the proof of the Hashin and Shtrikman variational principle [2, 30], have a constant sign. The only justification for choosing for  $L_C$  [7, 8] the Voight or Reiss estimates of  $L^*$  [8] is the fact that specific experimental data agree with the computed curves. In addition, the choice between the Voight or Reiss estimates is made on the basis of component by component comparison of the tensors of the elastic moduli of the components (and not their corresponding quadratic forms), and this leads to ambiguous results, even for isotropic materials.

In the well-known method of conditional moments [7, 8] it is assumed that the deformation fields are homogeneous not only inside the matrix but also inside the inclusions. The widely used "quasicrystalline" approximation (3.5) is used, and in obtaining specific estimates of  $L^*$  identically oriented inclusions consisting of a single material are usually considered. Each of these assumptions is stronger than the analogous assumptions in the EFM. For this reason, the single-particle EFM approximation (5.6)-(5.8) includes as a particular case the results found by the method of conditional moments [7, 8]. In [7, 8] the shape of the inclusions is taken into account via a prescribed anisotropy of the conditional distribution function  $\varphi(v_j|v_j; v_i)$ . For equally probable orientation of the inclusions it is possible to obtain an isotropic function  $\varphi$  [3] and the estimate of the effective modulus  $L^*$  will be invariant with respect to the shape of the inclusions. This result can be avoided by taking into account directly the shape of the inclusions via the tensors  $P$ , as done in [2] on the basis of a variational principle. For identically oriented ellipsoidal inclusions the results of [2, 8] are equivalent.

In the effective-method (method of self-consistency) [3] it is assumed that  $L_C = L^*$  and the particular case of the "quasicrystalline" approximation (3.5) is considered, which is equivalent to the assumption  $Y_{ij} = I \delta_{ij}$  in Eq. (5.6). In the singular-approximation method [3], which is invariant with respect to the shape of the inclusions, the operator in the general Eq. (1.3) with the kernel  $U$  is replaced by a constant tensor



$$g^s = \int U^s(x) dx$$

where  $U^S$  is the singular component of  $U$  [3]. This automatically implies a number of strong assumptions:  $P_i \equiv -g^S$ , equalities (2.3) are satisfied, the fields of the deformations in the components are homogeneous, the "quasicrystalline" approximation (3.5), and the functions  $\varphi(v_j|v_i; v_i)$  are isotropic. For this reason, the effective-medium method and the singular-approximation method [3] are also consequences of the EFM.

7. Remark. We now analyze the assumptions of the EFM and their generalizations. The assumption of the hypothesis H1 that the inclusions are ellipsoidal was used only in order to convert the integral Eq. (2.1) into an algebraic equation, since the tensor  $\langle U(x-y) \rangle_i$  is apparently homogeneous for  $x, y \in v_i$  only for an ellipsoid [13]. It can be assumed that in part of the region  $v_i \subset v_i$ ,  $M_1(x)$ ,  $\alpha(x) = 0$ , i.e., it is sufficient to include a real nonellipsoidal inclusion  $v_i \setminus v_i^1$  into an ellipsoid, possibly of smaller volume, and call it the inclusion  $v_i$ . The further scheme for calculating the tensors  $A$  and  $C$  (2.2) and  $L^*$  (1.5) remains the same, but the prescribed conditional distribution functions  $\varphi(v_j|v_i, \dots, v_n)$  will have a larger correlation well  $v_j^0$  than in a real composite material. This will result in underestimation of the computed values of  $L^*$  for inclusions which are more rigid than the matrix and overestimation in the opposite case.

The assumption that  $\bar{\varepsilon}_i(x)$  for  $x \in v_i$  is homogeneous was required in order to make it easier to solve the algebraic systems (2.1) and (3.1), which, in principle, can also be solved for a polynomial function  $\bar{\varepsilon}_i(x)$ ,  $\langle \hat{\varepsilon}(x) \rangle_{1, \dots, n}^i$ . Then, for example, the tensor  $A_i = \sum B_i^j$  (summation over  $j = 0, 1, \dots$ ) (2.2), where the index  $j$  takes into account the effect of the term of degree  $j$  in the polynomial  $\bar{\varepsilon}_i(x)$ . Similarly, in analyzing system (3.2)  $U(x_i - y)$  must be expanded in a Taylor series around  $x_j$  and the problem must be solved for a finite number of ellipsoids, as done in [31].

We note that in order to solve system (1.11), in any case, for homogeneous inclusions, it is not necessary to introduce intermediate concepts - effective fields  $\bar{\varepsilon}(x_i|x_i, \dots, x_n)$ ,  $\langle \hat{\varepsilon}(x) \rangle_{1, \dots, n}^i$ . System (1.11) is linear in the fields  $\langle \varepsilon(x) \rangle_{x_1, \dots, x_n}$ , and on closure [8], analogous to the hypothesis H2, it becomes finite and can be solved by the methods of linear algebra. This scheme is implemented by the method of conditional moments [7, 8]. According to the EFM, information about the geometric and mechanical characteristics is given by the tensors (2.2) and the field  $\langle \bar{\varepsilon}(x) \rangle_{1, \dots, n}^i$ , in contrast to  $\langle \varepsilon(x) \rangle_{x_1, \dots, x_n}^i$ , is weakly nonuniform. This is why, as noted in [17, 19], even rough assumptions about the structure of the effective field (5.1) make it possible to obtain correct results. In order to decrease the volume of calculations the tensor  $R_i$  and  $F_i$  can be replaced, everywhere in Eqs. (4.1) and the matrix  $Z$  (3.3), by their averages over the possible orientations of the inclusions  $v_i$ ; the analogous procedure by the method of [7, 8] is difficult to implement.

8. Analysis of Regular Structures. Highly efficient numerical methods have now been developed for calculating the effective moduli and local stresses in composite materials with periodic structure [32]. This can serve as a test for the accuracy of EFM. We now consider a periodic set  $X$  of ellipsoidal nonuniformities with identical shape, orientation, and mechanical properties. We represent the distribution of the particle centers in space as a sequence of vectors of a space lattice  $x_m = e_i m_i$ , where  $m_i$  ( $i = 1, 2, 3$ ) are positive integers, and  $e_i$  ( $i = 1, 2, 3$ ) are vectors oriented along the edges of a parallelepiped and are equal in modulus to the lengths of the edges. Then, for  $L_c = L^{(0)}$  formula (6.1) has the form

$$L^* = L^{(0)} + nR \left( I - P(w) Rn - \sum_i' S(x_j - x_i) Rn \right)^{-1}. \quad (8.1)$$

It was assumed that  $x_j$  coincides with the center of the region  $w$ , containing a quite large number of inhomogeneities (this will be specified quantitatively below). The summation in Eq. (8.1) extends over all  $x_i \in w$  and  $x_j \neq x_i$ . Under the assumption (4.3), employed in [23, 24], expression (8.1) changes to

$$L^* = L^{(0)} + nR \left( I - P(w) Rn - \sum_i' U(x_j - x_i) Rn \right)^{-1}. \quad (8.2)$$

TABLE 1

Method	c								
	0,2			0,4			0,5		
	$\mu^{(1)} = 0$								
	$L_{1111}^*$	$L_{1122}^*$	$L_{1212}^*$	$L_{1111}^*$	$L_{1122}^*$	$L_{1212}^*$	$L_{1111}^*$	$L_{1122}^*$	$L_{1212}^*$
[33]	2,26	0,83	1,33	1,48	0,42	0,94	1,13	0,26	0,76
(8.1)	2,26	0,83	1,28	1,46	0,45	0,77	1,14	0,33	0,60
(8.2)	2,21	0,85	1,35	1,59	0,38	0,42	1,35	0,22	0,01
	$\mu^{(1)} = 1000$								
[34]	5,30	1,92	2,90	8,80	2,26	4,48	17,08	2,44	6,50
(8.1)	5,28	1,91	2,90	8,64	2,40	4,28	11,1	2,87	5,80
(8.2)	5,53	1,77	2,82	-16,9	15,2	7,24	2,48	7,21	8,16

For simple cubic packing of spherical inclusions the tensor of effective moduli  $L^*$  is characterized by three elastic moduli. Table 1 gives for a porous medium ( $v^{(0)} = 0.3$ ,  $\mu^{(0)} = 1$ ,  $\mu^{(1)} = 0$ ), hard inclusions ( $v^{(0)} = v^{(1)} = 0.3$ ,  $\mu^{(1)}/\mu^{(0)} = 1000$ ,  $\mu^{(0)} = 1$ ), and a number of values of the volume concentration of inhomogeneities the values of  $L_{1111}^*$ ,  $L_{1122}^* = L_{2233}^* = L_{3311}^*$ ,  $L_{1212}^* = L_{2323}^* = L_{3131}^*$  computed by analytical methods [33, 34] and formulas (8.1) and (8.2).

According to the table, the error of the EFM (8.1) is maximum for hard nonuniformities with  $c = 0.5$  and does not exceed 15%. The calculation, by the approximate variant of the EFM (8.2), gives contradictory results for  $c > 0.35$ : the component  $L_{1111}^*$  oscillates around zero as  $c$  increases.  $L^*$  was calculated from formulas (8.1) and (8.2) for a spherical region  $w$  with  $\text{diam } w = 7|e_1|$ , containing three layers of spheres around a distinguished inclusion  $v_j$ ; for  $\text{diam } w = 5|e_1|$  (two layers) and  $\text{diam } w = 3|e_1|$  (one) the estimates of  $L^*$  (8.1) differ from those presented in Table 1 by 1.7 and 15%, respectively, i.e., an ensemble of nonuniformities with two layers of spheres can already be considered as representative and the principle of locality holds [21]. We note that an indirect justification of the accuracy of the single-particle EFM with the help of estimates of  $L^*$  for regular structures [24] is quite doubtful. Indeed, in this case, for random structures the terms in the summation in (8.1) and (8.2) are equal to zero, while for regular structures the values of these sums are comparable to  $P(w)Rn$ .

In conclusion we note that the advantage of the particular variants of EFM over different methods has also been demonstrated in comparison with experimental data [4-6, 35, 36] and existing analytical solutions for a regular system of collinear cracks in a plane [37].

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